Potential for inert adjoint scalar field in SU(2) Yang-Mills thermodynamics

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Abstract

We point out that the uniqueness of the potential for an inert adjoint scalar field describing spatially averaged, topological, BPS saturated, and stable field configurations, which are relevant for the ground-state structure of the thermodynamics of an SU(2) Yang-Mills theory being in its deconfining phase, follows without invoking detailed microscopic information.

1 Introduction

The potential importance of topological field configurations in generating a finite correlation length in the dynamics of thermalized, nonabelian gauge fields [1] was emphasized a long time ago [2]. In [3] the gauge groups SU(2) and SU(3) have been investigated: BPS saturated and stable (trivial holonomy) configurations of topological charge modulus $|Q| = 1$, Harrington-Shepard solutions [4], were integrated into an adjoint scalar field ϕ in the deconfining phase. Being a background to the coarse-grained dynamics of $Q = 0$ fluctuations the field ϕ prevents the infrared catastrophe of thermal perturbation theory from taking place by means of the adjoint Higgs mechanism. At the same time, the field ϕ acting as an ultraviolet cutoff in the coarse-grained theory, implies a rapidly converging loop expansion [3, 5].

In this note we would like to show that the dynamics of the field ϕ essentially follows by assuming its BPS saturation: No explicit consideration of the coarse-graining process over the relevant topological configurations in the underlying theory is needed to derive ϕ's potential $V(|\phi|^2) = \Lambda^6/|\phi|^2$. We also point out that the temperature dependence of the gluon condensate as obtained in the effective theory is in qualitative agreement with the lattice result of Ref. [6].

2 Gauge invariance, BPS saturation, and inertness

In the deconfining phase of SU(2) Yang-Mills thermodynamics the adjoint field ϕ, describing part of the thermal ground state, emerges from a spatial average over (anti)selfdual fundamental field configurations in euclidean spacetime where the time coordinate is compactified as $0 \le \tau \le \beta \equiv 1/T$. Without the need to perform the average explicitely the field ϕ enjoys the following properties as a consequence: (i) Since ϕ is obtained by a spatial coarse-graining over noninteracting, *stable*, BPS saturated field configurations (topology changing energy and pressure free fluctuations) it is itself BPS saturated. (ii) Originating from periodic-in-τ field configurations (in a given gauge) it is itself periodic. (iii) The gauge invariant modulus $|\phi|$ does not depend on spacetime (trivial expansion into Matsubara frequencies due to coarse-graining over energy and pressure free fluctuations).

We will now show that conditions (i)-(iii) fix the potential V for the field $\phi \equiv \phi^a(\tau)\,\lambda^a$ ($\mathrm{tr}\lambda_a\lambda_b = 2\delta_{ab}$, $a = 1, 2, 3$) uniquely when working with a canonical kinetic term in its euclidean Lagrangian density $\mathcal{L}_\phi$ [1]:

$$\mathcal{L}_\phi = \mathrm{tr}\,\left((\partial_\tau\phi)^2 + V(\phi^2)\right) \tag{1}$$

[1]As long as this term contains two powers of time derivatives this is not a constraint on generality due to (iii). Moreover, although we ignore the connection to the microscopic physics in the present work we surely can make an appropriate choice of gauge such that the coarse-graining over *noninteracting* topological defects generates $A_\mu = 0$ on the macroscopic level.

Since the coarse-graining is over exact solutions to the Yang-Mills equations the emerging field ϕ must minimize the effective action. Thus ϕ satisfies the Euler-Lagrange equations subject to Eq. (1):

$$\partial_\tau^2 \phi^a = \frac{\partial V(|\phi|^2)}{\partial |\phi|^2} \phi^a \leftrightarrow \partial_\tau^2 \phi = \frac{\partial V(\phi^2)}{\partial \phi^2} \phi \,, \tag{2}$$

where $|\phi| \equiv \sqrt{\frac{1}{2}\,\mathrm{tr}\,\phi^2}$. The gauge invariance of the potential $V = V(|\phi|^2)$ (central potential) in Eq. (2) implies that the solution has to describe motion in a plane of the three-dimensional vector space spanned by the Lie-algebra valued generators of SU(2) Yang-Mills theory[2].

Without restriction of generality (a global gauge choice) we choose the plane $(\phi^1, \phi^2, 0)$. Thus the solution takes the following form:

$$\phi = |\phi|\, \lambda_1 \exp(i\lambda_3 \theta(\tau)) = |\phi| \left(\lambda_1 \cos(\theta(\tau)) + \lambda_2 \sin(\theta(\tau))\right) \,, \tag{3}$$

or in components

$$(\phi^1, \phi^2, \phi^3) = |\phi| \left(\cos(\theta(\tau)),\, \sin(\theta(\tau)),\, 0\right) . \tag{4}$$

According to (ii) the function $\theta(\tau)$ needs to satisfy the following condition

$$\theta(\tau + \beta) = \theta(\tau) + 2\pi n \,, \tag{5}$$

where n is an integer. Finally, condition (i) implies the vanishing of the (euclidean) energy density $\mathcal{H}_E(\phi)$:

$$\mathcal{H}_E(\phi) = \mathrm{tr}\left((\partial_\tau \phi)^2 - V(\phi^2) \right) = 2\left((\partial_\tau \phi^a)^2 - V(|\phi|^2) \right) = 0 \quad , \quad \forall \beta \,. \tag{6}$$

Substituting Eq. (3) into Eq. (6) we have

$$|\phi|^2 \left(\partial_\tau \theta(\tau)\right)^2 - V(|\phi|^2) = 0 \,. \tag{7}$$

According to (iii) the potential $V(|\phi|^2)$ does not depend on τ. As a consequence of Eq. (7), we then have $\partial_\tau \theta(\tau) = \text{const}$. Together with Eq. (5) this yields:

$$\theta(\tau) = \frac{2\pi}{\beta} n \tau \tag{8}$$

up to an inessential constant phase (global gauge choice). Notice that the case $n = 0$ is excluded if we impose that[3] $V \neq 0$. Now Eq. (8) implies that $\partial_\tau^2 \theta(\tau) = 0$, and thus we obtain from Eqs. (3) and (2) that

$$\left(\partial_\tau \theta(\tau)\right)^2 = -\frac{\partial V(|\phi|^2)}{\partial |\phi|^2} \,. \tag{9}$$

[2]The angular momentum is a constant of motion in a central potential.

[3]In a similar way, the case $Q = 0$ is excluded for BPS saturated, microscopic field configurations if we insist on a nonvanishing action.

Eliminating $(\partial_\tau\theta(\tau))^2$ from Eqs. (7) and (9) we have:

$$\frac{V(|\phi|^2)}{|\phi|^2} = -\frac{\partial V(|\phi|^2)}{\partial\,|\phi|^2}\,. \tag{10}$$

Notice that Eq. (10) is valid for all values of β. The unique solution to the first-order differential equation (10) is

$$V(|\phi|^2) = \frac{\Lambda^6}{|\phi|^2} \tag{11}$$

where the mass scale Λ enters as a constant of integration. On dimensional grounds Λ has to appear with the sixth power. We interprete Λ as the Yang-Mills scale which, however, is not operational on the level of BPS saturated dynamics, see below[4].

Let us now determine the modulus $|\phi|$. By inserting the potential $V(|\phi|^2) = \Lambda^6/\,|\phi|^2$ and Eq. (8) into Eq. (7) we have

$$|\phi| = \sqrt{\frac{\Lambda^3}{2\pi|n|\,T}}\,. \tag{12}$$

This implies that the field ϕ vanishes in a power-like way with increasing temperature. The value of the integer n can not be determined within the macroscopic approach we have applied to deduce ϕ's potential. Microscopically, one observes that the definition of ϕ's phase does only allow for the contribution of Harrington-Shepard solutions [4] of topological charge modulus $|Q| = 1$ which implies that $n = \pm 1$ [3].

Finally, we point out the inertness of the field ϕ. According to Eqs. (11) and (1) the square of the mass M_ϕ of potential (radial) fluctuations $\delta\phi$ is given as (setting $|n| = 1$)

$$M_\phi^2 = 2\,\left.\frac{\partial^2 V}{\partial|\phi|^2}\right|_{|\phi|=\sqrt{\frac{\Lambda^3}{2\pi\,T}}} = 48\,\pi^2\,T^2\,. \tag{13}$$

Thus $\frac{M_\phi^2}{T^2} = 48\,\pi^2 \gg 1$, and no thermal excitations exist. On the other hand, we have $\frac{M_\phi^2}{|\phi|^2} = 12\,\lambda^3$ where $\lambda \equiv \frac{2\pi T}{\Lambda}$. For $\lambda \gg 1$ one has that $\frac{M_\phi^2}{|\phi|^2} \gg 1$. In practice, $\lambda > \lambda_c = 13.867$, see [3]. Since $|\phi|$ is the maximal resolving power allowed in the effective theory we conclude that quantum fluctuations of the field ϕ do not exist.

3 Topologically trivial fluctuations

3.1 Effective Lagrangian and ground state

For the reader's convenience we briefly repeat the derivation of [3] leading to the complete ground-state description of SU(2) Yang-Mills thermodynamics in its deconfining phase.

[4]On this level the energy-momentum tensor vanishes.

If topological fluctuations were absent then renormalizability [7] would assure that the action of the fundamental theory is form-invariant under the applied spatial coarse-graining. Since the topological part is integrated into an inert field ϕ this still holds true for the part of the effective action induced by $Q = 0$-fluctuations a_μ. We thus are confronted with the following, gauge invariant effective Lagrangian for the dynamics of coarse-grained $Q = 0$ fluctuations a_μ subject to the background ϕ:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}\left[a_\mu\right] = \text{tr}\,\left(\frac{1}{2}\,G^E_{\mu\nu}G^{\mu\nu}_E + (D_\mu\phi)^2 + \frac{\Lambda^6}{\phi^2}\right)\,, \tag{14}$$

where

$$\begin{aligned} G^{\mu\nu}_E &= \partial_\mu a_\nu - \partial_\nu a_\mu - ie\left[a_\mu, a_\nu\right] = G^{a,\mu\nu}_E\frac{\lambda_a}{2}\,, \\ a_\mu &= a^a_\mu\frac{\lambda_a}{2}\,, \qquad D_\mu\phi = \partial_\mu\phi - ie[a_\mu,\phi]\,, \end{aligned} \tag{15}$$

and e denotes an *effective* gauge coupling. According to Eq. (14) the equation of motion for the field a_μ is:

$$D_\mu G^{\mu\nu}_E = ie\left[\phi, D_\nu\phi\right]\ . \tag{16}$$

This is solved by the pure-gauge configuration $a_\mu = a^{gs}_\mu$ given as

$$a^{gs}_\mu = \mp\delta_{\mu 4}\frac{2\pi}{\beta}\frac{\lambda_3}{2} = \frac{i}{e}\left(\partial_\mu\Omega\right)\Omega^\dagger \text{ with } \Omega = e^{\pm i\frac{2\pi}{\beta}\tau\frac{\lambda_3}{2}} \Rightarrow D_\nu\phi = 0 \tag{17}$$

The entire ground state thus is described by the a^{gs}_μ, ϕ implying a ground-state pressure P^{gs} and energy density ρ^{gs} given as $P^{gs} = -4\pi\Lambda^3 T = -\rho^{gs}$: The inclusion of gluon fluctuations, which contribute to the dynamics of the ground-state, by virtue of $a_\mu = a^{gs}_\mu$ after coarse-graining shifts the vanishing results, obtained from BPS saturated configurations alone, to finite values. This makes the Yang-Mills scale Λ (gravitationally) visible. Turning to propagating fluctuations δa_μ in the effective theory it is advantageous to work in unitary gauge.

3.2 Unitary gauge and Higgs mechanism

By performing a gauge rotation[5] $U = e^{-i\frac{\pi}{4}\lambda_2}\Omega$ we have that $a^{gs}_\mu \rightarrow U a^{gs}_\mu U^\dagger - \frac{i}{e}\left(\partial_\mu U\right)U^\dagger = 0$ and $\phi = \lambda_3\,|\phi|$. This is the unitary gauge. The field strength $G^{\mu\nu}_E$ and the covariant derivative $D_\mu\phi$ are functionals of the fluctuations δa_μ only. We have

$$\mathcal{L}^{u.g.}_{\text{eff}} = \mathcal{L}\left[\delta a_\mu\right] = \frac{1}{4}\left(G^{a,\mu\nu}_E[\delta a_\mu]\right)^2 + 2e^2\,|\phi|^2\left(\left(\delta a^{(1)}_\mu\right)^2 + \left(\delta a^{(2)}_\mu\right)^2\right) + 2\frac{\Lambda^6}{|\phi|^2}\,. \tag{18}$$

[5] Notice that U is smooth and antiperiodic. One can introduce a center jump to make it periodic by sacrificing its smoothness [3]. However, the associated electric center flux does not carry any energy or pressure and the periodicity of effective gluon fluctuations is maintained. These are the physical reasons why the transformation to unitary gauge is admissible.

Fluctuations $\delta a_\mu^{(1,2)}$ are massive in a temperature dependent way while the mode $\delta a_\mu^{(3)}$ remains massless representing the fact that SU(2) is broken to its subgroup U(1) by the field ϕ. One has

$$m^2 = m_1^2 = m_2^2 = 4e^2 \left|\phi\right|^2 \,, \qquad m_3^2 = 0\,. \tag{19}$$

3.3 Energy density, pressure and running coupling

From the effective Lagrangian (18) we derive the energy density and the pressure on the one-loop level[6]

$$\rho = \rho_3 + \rho_{1,2} + \rho_{gs}\,, \quad p = p_3 + p_{1,2} + p_{gs}\,, \tag{20}$$

where the subscript 1,2 is understood as a sum over the two massive modes. Explicitely, we have:

$$\rho_3 \;=\; 2\frac{\pi^2}{30}T^4,\;\; \rho_{1,2} = 6\int_0^\infty \frac{k^2 dk}{2\pi^2}\frac{\sqrt{m^2+k^2}}{\exp(\frac{\sqrt{m^2+k^2}}{T})-1},\;\; \rho_{gs} = 2\frac{\Lambda^6}{|\phi|^2} = 4\pi\Lambda^3 T\,. \tag{21}$$

$$p_3 \;=\; 2\frac{\pi^2}{90}T^4,\;\; p_{1,2} = -6\,T\int_0^\infty \frac{k^2 dk}{2\pi^2}\ln\left(1-e^{-\frac{\sqrt{m^2+k^2}}{T}}\right)\,,\;\; p_{gs} = -\rho_{gs}\,. \tag{22}$$

The effective coupling constant e is a function of the temperature $e = e(T)$, and so is m. The function $e = e(T)$ is deduced by requiring the validity of the Legendre transformation

$$\rho = T\frac{dp}{dT} - p. \tag{23}$$

in the effective theory.

By substituting the equations (20) into (23) we obtain:

$$4\pi\Lambda^3 = -6D(m)\frac{dm(T)}{dT},\;\; D(m) = \int_0^\infty \frac{k^2 dk}{2\pi^2}\frac{m}{\sqrt{m^2+k^2}}\frac{1}{e^{\frac{\sqrt{m^2+k^2}}{T}}-1}\,. \tag{24}$$

Solving the differential equation (24) and inverting the solution, the function $e(T)$ follows by virtue of Eq. (19).

Eq. (24) is of first order. Thus a boundary condition needs to be prescribed. It was shown in [3] that the evolution at low temperature decouples from the boundary physics at high temperature. That is, there exists a low-temperature attractor to the evolution. This attractor is characterized by a logarithmic pole, $e \sim -\log(T - T_c)$ where $T_c = 13.867\frac{\Lambda}{2\pi}$, signalizing the presence of a phase transition, and by a plateau $e \equiv 8.89$ for T sufficiently larger than T_c, indicating magnetic charge conservation for screened monopoles.

[6] This is accurate on the 0.1%-level [3].

3.4 Gluon condensate

A further application of the present approach is possible. The gluon condensate is defined as

$$\frac{1}{32\pi^2 g^2}\left\langle F^a_{\mu\nu}F^{a,\mu\nu}\right\rangle(\mu) \tag{25}$$

where $\langle ...\rangle$ denotes the ground-state expectation value and g and $F_{\mu\nu}$ are the *fundamental* gauge coupling and field strength, respectively. Notice that the quantity in definition (25) depends on the renormalization scale μ. In the effective theory we have $\mu = |\phi|$. The expectation in (25) vanishes at any fixed order in perturbation theory [9]. Nonperturbatively, the gluon condensate does not vanish: It was related to instanton physics at $T = 0$ in [8, 9].

In euclidean spacetime the gluon condensate is just the average action density:

$$\frac{1}{32\pi^2 g^2}\left\langle F^a_{\mu\nu}F^{a,\mu\nu}\right\rangle = \frac{1}{8\pi^2}\left\langle \mathcal{L}^{\rm euc}_{\rm YM}\right\rangle . \tag{26}$$

The finite-temperature coarse-graining process of our approach recasts the Yang-Mills Lagrangian $\mathcal{L}^{\rm euc}_{\rm YM}$ of Eq. (26) into the effective Lagrangian $\mathcal{L}_{\rm eff}$ of Eq. 14). In $\mathcal{L}_{\rm eff}$ the (thermal) ground state is described by the two fields a^{gs}_μ, ϕ, see Sec. 3.1. The temperature dependent gluon condensate can be directly calculated as

$$\frac{1}{32\pi^2 g^2}\left\langle F^a_{\mu\nu}F^{a,\mu\nu}\right\rangle = \frac{1}{8\pi^2}\mathcal{L}_{\rm eff}[a^{b.g.}_\mu] = \frac{1}{4\pi^2}\frac{\Lambda^6}{|\phi|^2} = \frac{1}{2\pi}\Lambda^3 T,\ T > T_c = \frac{13.867}{2\pi}\Lambda. \tag{27}$$

The ground-state expectation of $\mathcal{L}_{\rm eff}[a^{b.g.}_\mu]$ is given by the potential $2\Lambda^6/|\phi|^2$. Integrating out thermal and quantum fluctuations in the effective theory would not contribute to the gluon condensate if the associated loop expansion would be exact at a finite, maximal number of loops. Modulo one-particle irreducible (1PI) resummations this is expected to happen [5]. Since 1PI resummations do not significantly alter the dispersion laws of the associated modes for a large part of their momentum spectrum only a tiny correction (typically 0.1% at maximum) is introduced to correct the ground-state part of the average over the effective action by fluctuating modes. For our comparison with lattice data it is hence sufficient to neglect fluctuating modes in calculating the average action density. Our approach predicts a gluon condensate which increases linearly with temperature for $T > T_c$ as $\frac{1}{2\pi}\Lambda^3 T$ ($T \geq T_c \equiv 11.65\,\frac{\Lambda}{2\pi}$) [3]. This is in qualitative agreement with the lattice result in Ref. [6].

4 Conclusions

In this note we have derived the potential $V(|\phi|^2) = \Lambda^6/|\phi|^2$ for an inert, adjoint scalar field ϕ by solely assuming its origin to be a spatial average over noninteracting, BPS saturated topological field configurations in the underlying theory: SU(2)

Yang-Mills thermodynamics being in its deconfining phase. That is, no detailed microscopic information on these configurations other than their stability and BPS saturation is needed to derive the potential for the effective field ϕ. The conceptually interesting implication of our present work is that the Yang-Mills scale Λ emerges as a constant of integration: Λ's existence needs not be assumed as in [3]. For our presentation to be selfcontained we have repeated the derivation of the effective action, involving the field ϕ as a background, for the coarse-grained, topologically trivial fluctuations. We also have pointed out that the (linear) temperature dependence of the gluon condensate agrees with that found in lattice simulations.

There is a host of applications of SU(2) Yang-Mills thermodynamics in particle physics [10] and cosmology [11].

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